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The Toroidal Thickness of the Symmetric Quadripartite Graph

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The toroidal thickness $t_1(G)$ of a graph G is the minimum value of k for which G is the edge-union of k graphs each embeddable on a torus. It is shown that $t_1(K_{4(n)}) = [\frac{1}{2}(n+1)]$.

The toroidal thickness $t_1(G)$ of a graph G is the minimum value of k for which G is the edge-union of k graphs each of which is embeddable on a torus. The toroidal thickness of the complete graph K_n was found by Ringel [6], and independently by Beineke [3] who also found $t_1(K_{n,n})$. In a recent paper [1] the toroidal thickness of G_n, the graph obtained from K_n by removing a hamiltonian cycle, was found, as well as $t_1(K_{n(3)})$ for many values of n.

In this paper we show $t_1(K_{4(n)}) = [\frac{1}{2}(n+1)]$ for all n. As in [5], the graph $K_{4(n)} = K_{n,n,n,n}$ can be described as the graph with vertices $0, 1, \ldots, 4n-1$, in which vertex i is joined to vertex j by an edge if and only if $i \not\equiv j \pmod 4$. In particular, vertex 0 is joined to all other vertices except those labelled by multiples of 4. The Euler formula shows that a graph on $4n$ vertices embedded on a torus can have at most $12n$ edges, this number being attained only when the embedding triangulates the torus. Since $K_{4(n)}$ has $6n^2$ edges, it cannot be split up into fewer than $\frac{1}{2}n$ toroidal subgraphs. Clearly, all the subgraph embeddings can be triangulations only when n is even; thus $t_1(K_{4(n)}) \geqslant [\frac{1}{2}(n+1)]$.

THEOREM. $t_1(K_{4(n)}) = [\frac{1}{2}(n+1)]$.

It is sufficient to prove that $t_1(K_{4(n)}) = \frac{1}{2}n$ whenever n is even, since then

$$k+1 \leqslant t_1(K_{4(2k+1)}) \leqslant t_1(K_{4(2k+2)}) = k+1.$$

We put $n = 2m$, and make use of the following lemma.

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LEMMA. *The numbers* 1,..., $4m$, *excluding* 4, 8,..., $4m$, *can be put into* m *triples* a, b, c *such that* g.c.d.$(a, b, c, 8m) = 1$ *and either* $a + b = c$ *or* $a + b + c = 8m$.

Proof of Lemma. Clearly we require one even and two odd numbers in each triple. The construction of triples presented here was inspired by observing that Jungerman's paper [5] contains implicitly the fact that, if m is odd, a simple method of forming triples exists. Starting with 1, add 4 at each stage, replacing each x greater than $4m$ by $8m - x$, its negative in Z_{8m}. For example, if $m = 5$, we obtain the sequence

$$\begin{array}{cccccccccc} 1 & 5 & 9 & 13 & 17 & 19 & 15 & 11 & 7 & 3 \\ \multicolumn{2}{c}{6} & \multicolumn{2}{c}{18} & \multicolumn{2}{c}{2} & \multicolumn{2}{c}{14} & \multicolumn{2}{c}{10} \end{array}$$

where the required even numbers are obtained as the sum or difference of the odd numbers, or as the sum subtracted from $8m$. In each triple the odd numbers differ by a power of 2 so that their g.c.d. with (or without!) $8m$ must be 1.

Let $m = 2^k u$, where u is odd. The triples

$$\begin{array}{ccc} 1 & 2^{k+2} + 1 & 2^{k+2} + 2 \\ 3 & 2^{k+2} + 3 & 2^{k+2} + 6 \\ \vdots & \vdots & \vdots \\ 1 \cdot 2^{k+2} - 1 & 2 \cdot 2^{k+2} - 1 & 3 \cdot 2^{k+2} - 2 \\ & & \\ 2 \cdot 2^{k+2} + 1 & 3 \cdot 2^{k+2} + 1 & 5 \cdot 2^{k+2} + 2 \\ \vdots & \vdots & \vdots \\ 3 \cdot 2^{k+2} - 1 & 4 \cdot 2^{k+2} - 1 & 7 \cdot 2^{k+2} - 2 \end{array}$$

and further blocks of triples down to

$$\begin{array}{ccc} (u-3) \cdot 2^{k+2} + 1 & (u-2) \cdot 2^{k+2} + 1 & (2u-5) \cdot 2^{k+2} + 2 \\ \vdots & \vdots & \vdots \\ (u-2) \cdot 2^{k+2} - 1 & (u-1) \cdot 2^{k+2} - 1 & (2u-3) \cdot 2^{k+2} - 2 \end{array}$$

cover all the odd numbers up to $(u-1) \cdot 2^{k-2} - 1$; further, the even numbers, or their negatives in Z_{8m}, are precisely the required even numbers greater than 2^{k+2}. Each such triple a, b, c has a, b odd and $b - a = 2^{k+2}$, so g.c.d.$(a, b, c, 8m) = 1$. Finally, the triples

$$(u-1) \cdot 2^{k+2} - 1 + 2r \qquad u \cdot 2^{k+2} + 1 - 2r \qquad 2^{k+2} + 2 - 4r$$
$$(r = 1, \ldots, 2^k)$$

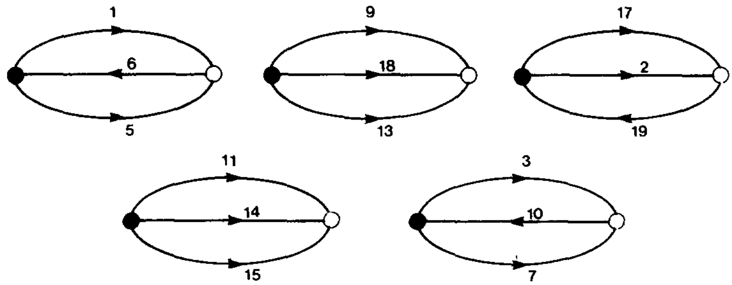

FIG. 1. $t_1(K_{4(10)}) = 5$.

cover the remaining numbers; the sum of the odd numbers is $8m - 2^{k+2}$, so that again g.c.d.$(a, b, c, 8m) = 1$. Thus the lemma is proved. ∎

Using the ideas in [1, 2] the theorem now follows by using the triples a, b, c, as currents on $\frac{1}{2}n$ current graphs of the type in Fig. 1 (which illustrates the case $n = 10$).

In conclusion, we observe that, since the graphs used are bipartite, the triangulations of the tori which result all have bichromatic dual. It therefore follows from [7] that for each even n we obtain a $(4n, 4n^2, 3n, 3; 0, 2)$ partially balanced incomplete block design (PBIBD) which splits into two $(4n, 2n^2, \frac{3}{2}n, 3; 0, 1)$ PBIBDs. These differ from the designs obtained by Garman in [4], since Garman's are derived from embeddings of $K_{4(n)}$ on different surfaces.

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